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EXACT A PRIORI MATCHING OF MIXED BOUNDARY CONDITIONS FOR SECOND--ETC(U)
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EXECUTIVE SUMMARY

of the attached paper entitled

"Exact A Priori Matching of Mixed Boundary Conditions for Second Order Elliptic Problems"

by

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The research results reported in the present paper represent a substantial advance in the direction of providing more efficient, cost-saving techniques for solving a wide class of commonly occurring two-dimensional boundary value problems. In previous papers ([5], [6]), it has been shown that it is possible to dramatically reduce the cost of solving two-dimensional problems by amalgating three formerly disparate problem-solving tools, namely:

1. Computer graphics (visual feedback)
2. Numerical analysis (scientific computing software)
3. Qualitative information (the analyst's experience and insight, and "weak" mathematical theorems).

More specifically, in [5], Gordon and Hall pointed out (via examples) the practical utility of such an amalgamation. The problems considered therein were, however, restricted to elliptic boundary value problems subject to Dirichlet boundary conditions, i.e., problems in which the function values are specified on the perimeter of the domain. That paper, as well

as [6], focused on the issue of contrasting the usual way of initializing an iterative numerical solution method with the proposed new technique which uses the so-called "blending-function methods" of interpolation to *a priori* exactly match the boundary conditions.

As one would intuitively expect, starting with what literally "looks like" (computer graphics) a good approximation reduces the computation (numerical analysis) time very substantially. If, in addition, an analyst is provided a mechanism for quantifying his experience-based knowledge (qualitative information) of the particular class of problems under study, the "exact" solution is almost in hand.

The Gordon/Kelly paper [6] extends these early results, involving only Dirichlet boundary conditions, to the rather general problem of satisfying "mixed linear boundary conditions," i.e., boundary conditions of the form: $\alpha F + \beta \frac{\partial F}{\partial n} = g$. The boundary conditions are, however, assumed to be "consistent." By this is meant that, at the corners of the region, the boundary conditions from either side "match."

The attached paper addresses the problem of inconsistently specified boundary conditions. In the simplest instance of Dirichlet conditions, this means that the function values do not match at the corners. Herein, we show how to actually construct bivariate functions which exactly match the above type mixed linear boundary conditions, even when they are inconsistently specified, cf. Section III. Moreover, software has been developed which numerically performs the necessary operator multiplications and constructs the singular functions needed to accommodate such inconsistent boundary conditions. This software will soon be available for general distribution.

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EXACT A PRIORI MATCHING OF MIXED BOUNDARY CONDITIONS FOR SECOND ORDER ELLIPTIC PROBLEMS

This paper is to appear in the *Proceedings of the Fourth International Symposium on Computer Methods for Partial Differential Equations* to be held on June 30 - July 2, 1981, at Lehigh University.

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ABSTRACT

In this paper we consider the problem of constructing bivariate functions which exactly match boundary conditions of the general form $\alpha F + \beta(\partial F/\partial n) = g(x,y)$ on the perimeter of the unit square. The reason for wishing to do this is that substantial savings in computation time can be realized in the subsequent solution of the discretized boundary value problem: If an iterative method is used to solve the discretized problem, beginning with a good initial approximation can dramatically reduce the number of iterations required to achieve convergence; if a direct solution method is used, a specified accuracy can be achieved with far fewer algebraic unknowns. Although attention herein is restricted to rectangular regions, the techniques developed can be straightforwardly extended to any rectangular polygon. The interpolation techniques which we develop for exact boundary matching are illustrated by several examples which are accompanied by perspective views of their graphs and by contour plots.

I. Background and Introduction

To set the stage for the main ideas of this paper, we begin by displaying the familiar bilinearly blended interpolant to *Dirichlet boundary conditions* on the perimeter of the unit square $S = [0,1]\times[0,1]$. Let $F(x,y)$ be a supposed primitive function from which the boundary conditions are extracted. Then, the *synthetic function* $U(x,y)$ which we construct via transfinite ("blending function") interpolation is given by the following (cf., [1], [2], [5]):

$$\begin{aligned} U(x,y) = \quad & (1-x)F(0,y) + xF(1,y) \\ &+ (1-y)F(x,0) + yF(x,1) \\ &- (1-x)(1-y)F(0,0) - (1-x)yF(0,1) \\ &- x(1-y)F(1,0) - xyF(1,1). \end{aligned} \tag{1.1}$$

It is easy to verify that $U = F$ on ∂S.

As an example of the use of (1.1), consider a primitive function F whose values along the four edges of the unit square are:

$$\begin{aligned} F(0,y) &= \sin(2\pi y) + 1.5 \qquad & F(1,y) &= y^2(y-1) + .5 \\ F(x,0) &= 1.5 - x^2 & F(x,1) &= (x-1)^2 + .5. \end{aligned} \tag{1.2}$$

The function U which interpolates these boundary conditions is given by (1.1) as:

$$\begin{aligned} U(x,y) = &(1-x)\sin(2\pi y) + xy(y^2-y-2) \\ &+ x^2(2y-1) + 1.5. \end{aligned} \tag{1.3}$$

The graph of U is shown in Fig. 1a, and a contour plot of U is given in Fig. 1b.

Bivariate and higher dimensional interpolation is most easily discussed in the formalism of projection operators ([1], [2], [5], [6]). For instance, the above expression (1.1) for U can be more succinctly expressed as the *Boolean sum* of the two elementary projectors P_x and P_y given by:

$$\begin{aligned} P_x[F] &= (1-x)F(0,y) + xF(1,y) \\ P_y[F] &= (1-y)F(x,0) + yF(x,1). \end{aligned} \tag{1.4}$$

By definition, the Boolean sum of two projectors (idempotent linear operators) is: $P_x \oplus P_y = P_x + P_y - P_xP_y$. *For the time being,* we shall assume that the primitive function F is continuous at the four corners of the unit square, in which case the projectors commute:

$$P_xP_y[F] = P_yP_x[F]. \tag{1.5}$$

(The main results of this paper are, as we shall soon discuss, concerned with problems involving corner singularities; in those cases, the relevant projectors do not commute.) In terms of these commutative projectors P_x and P_y, expression (1.1) for U is simply:

$$U = (P_x \oplus P_y)[F] = (P_y \oplus P_x)[F]. \tag{1.6}$$

Interpolation schemes of this type are known by several aliases including *Boolean sum interpolation, transfinite interpolation, and blending function interpolation.* Of these, the term transfinite comes closest to conveying the essence of this class of techniques. These methods are distinguished from classical finite dimensional interpolation schemes by the fact that they incorporate a nondenumerable number of scalar samples of F into the interpolant. More precisely, interpolation schemes of this class extract from the bivariate primitive function F *univariate samples* of F, not simply scalar samples. (Note that $P_x[F]$ and $P_y[F]$ individually and $(P_x \oplus P_y)[F]$ are transfinite interpolants, the product $P_xP_y[F] = P_yP_x[F]$ is merely the standard four parameter bilinear interpolant to the four corner values of F.)

Previous studies by Gordon and Hall [5] and Gordon and Kelly [6] have been aimed at demonstrating how, for continuous boundary conditions, the transfinite, bilinearly blended interpolant (1.1) can be employed to reduce the computational effort in obtaining numerical solutions to second order elliptic boundary value problems. In these two papers, the authors discuss the following general approach: First, use (1.1) to construct U, which exactly matches the given Dirichlet boundary conditions and thus reduces the original problem to one with homogeneous boundary conditions; then, examine the original problem for any additional information which may be inferred about the solution. Such auxiliary knowledge, although perhaps merely qualitative or heuristic, can often be used to advantage in improving upon the first approximation U obtained by simply matching the boundary conditions.

As an example, the solution to Laplace's equation must satisfy a Maximum Principle. If the initial estimate U does not, then there are simple ways of constructing functions V which vanish on ∂S and are such

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that U + V both matches the given boundary conditions and satisfies the Maximum Principle, cf., [5] and [6]. For the Poisson equation $\nabla^2 U = \rho$, the sign of ρ determines that the solution is (locally) either subharmonic or superharmonic [8], and this auxiliary information can be built into the exact boundary matching approximation U + V. In the actual testing of these ideas, we have found that *interactive computer graphics* is an almost indispensable aid.

Numerical experiments using the techniques suggested in [5] and [6] to obtain good first approximations with which to enter standard *iterative* linear system solvers demonstrated that very substantial reductions in total computational cost can be realized using such *preprocessing methods*. Inasmuch as, lacking any previously computed results, the standard initialization of an iterative scheme for solving large linear systems is to set all unknowns equal to zero (or some constant), it is no surprise that an exact boundary matching function which also incorporates readily available auxiliary information should produce a more rapidly convergent numerical solution.

What may be more surprising are the results reported by Mitchell, Marshall and Wait ([9], [10, pp. 174-175]), and by Rice [11]. Namely, that merely by reducing an elliptic problem with inhomogeneous boundary conditions to a problem with homogeneous conditions one is able to achieve a numerical solution of specified accuracy using far fewer algebraic unknowns. (Exact *a priori* matching of boundary conditions is tantamount to reducing the original problem to a problem whose solution — the "residual" -- must satisfy homogeneous boundary conditions.) Rice has observed this empirically for the collocation codes in the ELLPACK suite, and Marshall and Mitchell have reported this to be true in experiments contrasting standard bilinear finite elements with "exact boundary elements". For the potential flow problem ($\nabla^2 U = 0$) with a source at (.437,-k), the exact solution of which is U = log r where $r^2 = (x-.437)^2 + (y+k)^2$, Marshall and Mitchell obtained results indicating that for a "weak" singularity at (.437,-3), more than 256 standard bilinear elements are required with *inhomogeneous* boundary conditions to achieve the same (four-figure) accuracy as can be obtained with 16 elements if the boundary conditions are first homogenized. If the singularity is located at (.437,-.1), the comparison is roughly 256 elements to achieve three-figure accuracy with inhomogeneous conditions versus 64 with homogeneous, cf. Table 4, p. 175 of [10].

In brief, the development of methodologies and associated software preprocessors to *a priori* exactly match rather general boundary conditions, and thus permit their homogenization prior to discretization and numerical solution, promises considerable savings in total computational cost, whether the discrete linear system is solved by iterative or direct methods.

II. Transfinite Interpolation to Mixed (Consistent) Boundary Conditions

In this section, we consider rather general boundary conditions of the form $\alpha F + \beta(\partial F/\partial n) = g$ on the perimeter of the unit square. These boundary conditions are to be thought of as being associated with some second order elliptic boundary value problem and, without further mention, we shall assume that they are such as to guarantee that the problem is well-posed. In particular, this means that the solution must be "pinned" along at least one of the four edges, i.e., on at least one of the edges the function value itself must be specified.

In [6], Gordon and Kelly considered *mixed linear boundary conditions* of the form:

$$\begin{aligned}
\mathcal{L}_0[F] &= \alpha_0 F(0,y) + \beta_0 F_x(0,y) = g_0(y) && \text{along } x=0\\
\mathcal{L}_1[F] &= \alpha_1 F(1,y) + \beta_1 F_x(1,y) = g_1(y) && \text{along } x=1\\
\mathcal{M}_0[F] &= \bar\alpha_0 F(x,0) + \bar\beta_0 F_y(x,0) = h_0(x) && \text{along } y=0\\
\mathcal{M}_1[F] &= \bar\alpha_1 F(x,1) + \bar\beta_1 F_y(x,1) = h_1(x) && \text{along } y=1
\end{aligned} \tag{2.1}$$

in which the α_i, $\bar\alpha_i$, β_i and $\bar\beta_i$ are *constants*, and the boundary conditions are *consistent*, i.e.:

$$\mathcal{L}_i\mathcal{M}_j[F] = \mathcal{M}_j\mathcal{L}_i[F] \qquad (i,j = 0,1). \tag{2.2}$$

In the case of Dirichlet conditions, the *linear operators* $\mathcal{L}_i$ and $\mathcal{M}_j$ are just:

$$\begin{aligned}
\mathcal{L}_0[F] &= F(0,y) = g_0(y), & \mathcal{L}_1[F] &= F(1,y) = g_1(y)\\
\mathcal{M}_0[F] &= F(x,0) = h_0(x), & \mathcal{M}_1[F] &= F(x,1) = h_1(x)
\end{aligned} \tag{2.3}$$

and the consistency requirement simply means that the boundary conditions are *continuous* at the four corners:

$$\mathcal{M}_j[g_i(y)] = \mathcal{L}_i[h_j(x)] \qquad (i,j = 0,1). \tag{2.4}$$

<u>*Theorem*</u> (Gordon/Kelly): Let the $\mathcal{L}_i$ and $\mathcal{M}_j$ be as in (2.1) and define two projectors P_x and P_y as follows:

$$\begin{aligned}
P_x[F] &= \phi_0(x)\mathcal{L}_0[F] + \phi_1(x)\mathcal{L}_1[F]\\
P_y[F] &= \psi_0(y)\mathcal{M}_0[F] + \psi_1(y)\mathcal{M}_1[F],
\end{aligned} \tag{2.5}$$

where the functions ϕ_i and ψ_j satisfy the *cardinality conditions:*

$$\begin{aligned}
\mathcal{L}_i[\phi_k] &= \delta_{ik} \quad \text{for } i,k = 0,1\\
\mathcal{M}_j[\psi_\ell] &= \delta_{j\ell} \quad \text{for } j,\ell = 0,1.
\end{aligned} \qquad \text{(Kronecker Delta)} \tag{2.6}$$

Then, the function U obtained from the Boolean sum of P_x and P_y exactly satisfies all of the specified boundary conditions.

<u>*Proof:*</u> The function U is given by

$$\begin{aligned}
U &= (P_x \oplus P_y)[F]\\
&= \phi_0(x)\mathcal{L}_0[F] + \phi_1(x)\mathcal{L}_1[F] + \psi_0(y)\mathcal{M}_0[F] + \psi_1(y)\mathcal{M}_1[F]\\
&\quad - \phi_0(x)\psi_0(y)\mathcal{L}_0\mathcal{M}_0[F] - \phi_0(x)\psi_1(y)\mathcal{L}_0\mathcal{M}_1[F]\\
&\quad - \phi_1(x)\psi_0(y)\mathcal{L}_1\mathcal{M}_0[F] - \phi_1(x)\psi_1(y)\mathcal{L}_1\mathcal{M}_1[F].
\end{aligned} \tag{2.7}$$

The proof consists of a straightforward verification of the facts that $\mathcal{L}_i[U] = g_i(y)$ and $\mathcal{M}_j[U] = h_j(x)$. We have, for example:

$$\begin{aligned}
\mathcal{L}_0[U] &= \mathcal{L}_0[F] + \psi_0(y)\mathcal{L}_0\mathcal{M}_0[F] + \psi_1(y)\mathcal{L}_0\mathcal{M}_1[F]\\
&\quad - \psi_0(y)\mathcal{L}_0\mathcal{M}_0[F] - \psi_1(y)\mathcal{L}_0\mathcal{M}_1[F]\\
&= \mathcal{L}_0[F]\\
&= g_0(y)
\end{aligned} \tag{2.8}$$

in which we have used the *cardinality conditions* (2.6). To show that $\mathcal{M}_j[U] = \mathcal{M}_j[F] = h_j(x)$, we also use the consistency hypotheses (2.2). <u>Q.E.D.</u>

As an illustration of this result, consider a function F such that:

$$\mathcal{L}_0[F] = F_x(0,y) \qquad = \frac{-1.2}{(-1.2)^2 + (y-.9)^2}$$

$$L_1[F] = 6F(1,y) - F_x(1,y) = 6[\ln\sqrt{(-.2)^2 + (y-.9)^2} + 1.5] + \frac{.2}{(-.2)^2 + (y-.9)^2}$$

$$M_0[F] = F(x,0) + 2F_y(x,0) = \ln\sqrt{(x-1.2)^2 + (-.9)^2} - \frac{1.8}{(x-1.2)^2 + (-.9)^2} + 1.5 \qquad (2.9)$$

$$M_1[F] = F(x,1) = \ln\sqrt{(x-1.2)^2 + (.1)^2} + 1.5$$

It may be easily verified that the four functions:

$$\phi_0(x) = x - \frac{5}{6} \qquad \phi_1(x) = \frac{1}{6}$$
$$\psi_0(y) = y - 1 \qquad \psi_1(y) = 2 - y \qquad (2.10)$$

satisfy the necessary cardinality conditions: Simply apply the formulas for the L_i and M_i to these univariate "blending functions" and confirm the Kronecker delta properties. The corner values are $L_0M_0[F] = -1.387$, $L_0M_1[F] = -.828$, $L_1M_0[F] = -2.962$, and $L_1M_1[F] = 4.013$. Thus, the function U given by

$$\begin{aligned}U(x,y) = {} & (.833-x)\frac{1.2}{1.44 + (y-.9)^2} + \ln\sqrt{.04 + (y-.9)^2} \\ & + \frac{.1}{3(.04 + (y-.9)^2)} + \left[(y-1)\ \ln\sqrt{(x-1.2)^2 + .81}\right. \\ & \left. - \frac{1.8}{(x-1.2)^2 + .81}\right] + (2-y)\ln\sqrt{(x-1.2)^2 + 0.1} \\ & + 1.387(x-.833)(y-1) + .828(x-.833)(2-y) \\ & - 1.994(y-1) + .831(2-y) + 1.5 \end{aligned} \qquad (2.11)$$

satisfies the boundary conditions: $L_i[U] = L_i[F]$ and $M_j[U] = M_j[F]$ $(i,j = 0,1)$ on the perimeter of S. The graph of U is displayed in Fig. 2a, and its contour plot is depicted in Fig. 2b.

In the earlier work by Gordon and Kelly, the authors *assumed* the existence of blending function $\phi_i(x)$ and $\psi_j(y)$ which satisfy the requisite cardinality conditions (2.6). Here we show, for boundary conditions of the general form (2.1), how to actually *construct* the $\phi_i(x)$ and $\psi_j(y)$. In particular, we show that for any choice of the eight parameters α_i, β_i, $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ in (2.1), we can always find polynomials of degree three or less which satisfy (2.6).

Lemma: Let L_i and M_j, and the projectors P_x and P_y be defined as in the above theorem. Then, there exist polynomials of maximal degree three such that (2.6) holds.

Proof: We need carry out the proof for only the $\phi_i(x)$, since the $\psi_j(y)$ are constructed independently and analogously. To this end, suppose that ϕ_0 is cubic in x:

$$\phi_0(x) = a_0 + b_0x + c_0x^2 + d_0x^3. \qquad (2.12)$$

By applying the linear operators L_0 and L_1 to ϕ_0 and collecting terms, we obtain the linear system

$$\begin{aligned} L_0[\phi_0] &= \alpha_0 a_0 + \beta_0 b_0 &&= 1 \\ L_1[\phi_0] &= \alpha_1 a_0 + (\alpha_1+\beta_1)b_0 + (\alpha_1+2\beta_1)c_0 + (\alpha_1+3\beta_1)d_0 &&= 0 \end{aligned} \qquad (2.13)$$

for the determination of the polynomial coefficients a_0, b_0, c_0 and d_0. (Bear in mind that the constants α_i and β_i, which completely characterize L_0 and L_1, are known.) Clearly, since there are four unknowns and only two equations, this system is, in general, underdetermined. Our criterion for selecting one among the (in general) two-parameter family of solutions is to take that solution which corresponds to the minimal degree polynomial. This resolves all ambiquities except two:

1. If $\alpha_0(\alpha_1+\beta_1) - \alpha_1\beta_0 = 0$ *and* both of the following 2×2 submatrices are nonsingular:

$$\begin{bmatrix}\alpha_0 & 0 \\ \alpha_1 & (\alpha_1+2\beta_1)\end{bmatrix} \ \textit{and}\ \begin{bmatrix}\beta_0 & 0 \\ (\alpha_1+\beta_1) & (\alpha_1+2\beta_1)\end{bmatrix}. \qquad (2.14)$$

In this case, we solve for a_0 and c_0 from the first of these and set $b_0 = d_0 = 0$.

2. If all 2×2 submatrices are singular *except*

$$\begin{bmatrix}\alpha_0 & 0 \\ \alpha_1 & (\alpha_1+3\beta_1)\end{bmatrix} \ \textit{and}\ \begin{bmatrix}\beta_0 & 0 \\ (\alpha_1+\beta_1) & (\alpha_1+3\beta_1)\end{bmatrix}, \qquad (2.15)$$

then solve for a_0 and d_0 from the first of these and set $b_0 = c_0 = 0$.

Thus, assuming that cubic blending functions do exist, the above procedure produces a unique ϕ_0. The function ϕ_1 is obtained in a completely analogous fashion.

Now, we must show that the linear system (2.13) will always have *at least* one solution. To see this, consider the conditions under which all five of the nontrivial 2×2 submatrices of (2.13) are singular:

$$\begin{aligned} \alpha_0(\alpha_1+\beta_1) - \alpha_1\beta_0 &= 0 \\ \alpha_0(\alpha_1+2\beta_1) &= 0 \\ \alpha_0(\alpha_1+3\beta_1) &= 0 \\ \beta_0(\alpha_1+2\beta_1) &= 0 \\ \beta_0(\alpha_1+3\beta_1) &= 0. \end{aligned} \qquad (2.16)$$

The key to the proof of existence is the recognition that if $\alpha_i = 0$, then β_i cannot be zero, and vice versa ($i=0$ and 1). With this in mind, it is easy to show that the five equations of (2.16) cannot all hold simultaneously. Q.E.D.

As an example, consider the following *consistent* boundary conditions:

$$\begin{aligned} L_0[F] &= F(0,y) + F_x(0,y) &&= 1.5\pi e^{-y} + .2 \\ L_1[F] &= F(1,y) &&= .25e^{-(1+y)}\cos(\pi y) + .2 \\ M_0[F] &= 2F(x,0) + F_y(x,0) &&= .25e^{-x}(1+\sin 6\pi x) + .4 \\ M_1[F] &= 2F(x,1) - F_y(x,1) &&= .75e^{-(1+x)}(-1+\sin 6\pi x) + .4. \end{aligned} \qquad (2.17)$$

Here, $\alpha_0=1$, $\beta_0=1$, $\alpha_1=1$, $\beta_1=0$, $\tilde{\alpha}_0=2$, $\tilde{\beta}_0=1$, $\tilde{\alpha}_1=2$ and $\tilde{\beta}_1=-1$, so that, by following the above algorithm, we obtain for the blending functions:

$$\phi_0(x) = 1 - x^2 \qquad \phi_1(x) = x^2$$
$$\psi_0(y) = .5 + y^3 \qquad \psi_1(y) = -y^3 \qquad (2.18)$$

which yield the function:

$$U(x,y) = 1.5\pi(1-x^2)e^{-y} + .25x^2(e^{-(1+y)}\cos\pi y)$$

$$+ .25e^{-x}(1+\sin 6\pi x)(.5+y^3) \qquad (2.19)$$

$$- .75e^{-(1+x)}(-1+\sin 6\pi x)y^3 - 1.5\pi(1-x^2)(.5+y^3)$$

$$+ 4.5\pi e^{-1}(1-x^2)y^3 - .25e^{-1}x^2(.5+y^3)$$

$$- .75e^{-2}x^2y^3 + .2.$$

By applying the four linear operators L_i and M_j to this last expression, it can be confirmed that $L_i[U] = L_i[F]$ and $M_j[U] = M_j[F]$ $(i,j = 0,1)$, i.e., U *does* satisfy the requisite boundary conditions. The graph and contour plot of U are shown in Figs. 3a and 3b.

III. Transfinite Interpolation to Inconsistent Mixed Boundary Conditions

As a practical matter, boundary conditions for elliptic problems are quite frequently not consistently specified. By this we mean that, although the solution must be smooth (analytic) inside the problem domain, it may and often does have singularities (discontinuities) on the boundary. An elementary example of this is the textbook heat conduction problem of determining the equilibrium temperature distribution in a square plate, three sides of which are immersed in ice (0°C) and the fourth in steam (100°C). (Figs. 4a and 4b depict the graph of the solution and its contour plot.)

In simple instances such as this with Dirichlet boundary conditions, the analyst faced with solving the boundary value problem will undoubtedly be aware of the singular behavior at two of the corners since it is so conspicuous. As a rule, however, inconsistently specified mixed boundary conditions are not easily spotted. For instance, suppose that a Dirichlet condition $F(0,y) = g_0(y)$ is specified along the edge x=0 of the unit square and that along the edge y=0 the Neumann condition $F_y(x,0) = h_0(x)$ is given. In order for the solution to be smoothly continuous at the corner (0,0), it is necessary that these two conditions be consistent, i.e.:

$$\lim_{y\to 0} \frac{d}{dy} F(0,y) = \lim_{x\to 0} F_y(x,0)$$

$$h_0(x)\Big|_{x=0} = \frac{d}{dy} g_0(y)\Big|_{y=0}. \qquad (3.1)$$

The question of consistency or inconsistency is, of course, even more subtle for general boundary conditions of the form (2.1) above.

Fortunately, high quality numerical software for solving elliptic problems is sufficiently robust as to be able to accommodate even grossly inconsistent boundary conditions. Provided that a solution exists, by taking a sufficiently fine discretization (and perhaps employing some special tricks), the applied analyst can normally obtain a solution to whatever accuracy desired. This, however, is a computationally expensive procedure which can be better handled by *a priori* taking cognizance of the anticipated singular behavior near corners. This is the main goal of this section.

With the same notation as the previous section, *we now consider boundary operators L_i and M_j which do not commute:* $L_iM_j[F] \neq M_jL_i[F]$ $(i,j = 0,1)$. The noncommutativity of the L_i and M_j is what we mean by *inconsistent boundary conditions.* We still find it useful to consider the projection operators P_x and P_y of (2.5) in which the $\phi_i(x)$ and $\psi_j(y)$ are determined in precisely the same way as outlined in the previous section. Now, however,

$$P_xP_y[F] \neq P_yP_x[F], \qquad (3.1)$$

which has the important implication that the function $U = (P_x \oplus P_y)[F]$ will *not* satisfy the (inconsistent) boundary conditions. (Actually, $(P_x \oplus P_y)[F]$ does satisfy the conditions that $L_i[(P_x \oplus P_y)[F]] = L_i[F]$ $(i=0,1)$, but does not satisfy the other two conditions: $M_j[(P_x \oplus P_y)[F]] \neq M_j[F]$.)

In order to deal with corner inconsistencies, we develop a class of interpolants specifically designed for the purpose. At each corner (i,j) we construct a special function U_{ij} such that

$$\left.\begin{aligned} L_kM_\ell[U_{ij}] &= \delta_{ik}\delta_{j\ell}L_iM_j[F] \\ M_\ell L_k[U_{ij}] &= \delta_{ik}\delta_{j\ell}M_jL_i[F]. \end{aligned}\right. \qquad (3.2)$$

In words, for fixed i and j, the function U_{ij} vanishes under operation by any of the six *linear functionals* L_kM_ℓ and $M_\ell L_k$ $(k\neq i,\ \ell\neq j)$. When operated on by L_iM_j or M_jL_i, the result is $L_iM_j[F]$ or $M_jL_i[F]$, respectively.

The case of pure Dirichlet boundary conditions $(\alpha_0 = \alpha_1 = \bar\alpha_0 = \bar\alpha_1 = 1,\ \beta_0 = \beta_1 = \bar\beta_0 = \bar\beta_1 = 0)$ is the simplest to interpret. Suppose the boundary conditions are such that

$$\left.\begin{aligned} F(x,j)\Big|_{x=i} &= \lim_{x\to i} F(x,j) = L_iM_j[F] \\ F(i,y)\Big|_{y=j} &= \lim_{y\to j} F(i,y) = M_jL_i[F] \end{aligned}\right\} \quad (i,j=0,1) \qquad (3.3)$$

and $L_iM_j[F] \neq M_jL_i[F]$; cf., for example, Fig. 4a. The function U_{ij} which we shall construct will satisfy, for the case of Dirichlet conditions:

$$\left.\begin{aligned} \lim_{x\to i} U_{ij}(x,j) &= L_iM_j[F] = \lim_{x\to i} F(x,j) \\ \lim_{y\to j} U_{ij}(i,y) &= M_jL_i[F] = \lim_{y\to j} F(i,y) \end{aligned}\right\} \quad (i,j=0,1) \qquad (3.4)$$

and at the three corners other than (i,j), U_{ij} will vanish.

In the general case, suppose for the moment that we have the required functions $U_{ij}(x,y)$ which satisfy conditions (3.2). Let W be equal to the sum of these four corner functions:

$$W(x,y) = U_{00}(x,y) + U_{01}(x,y) + U_{10}(x,y) + U_{11}(x,y). \qquad (3.5)$$

Clearly, W satisfies the eight conditions:

$$L_iM_j[W] = L_iM_j[F],\ M_jL_i[W] = M_jL_i[F]\ (i,j=0,1). \qquad (3.6)$$

From this, we draw the important conclusion that, by virtue of the linearity of the operators L_i and M_j:

$$\left.\begin{aligned} L_iM_j[F - W] &= 0 \\ M_jL_i[F - W] &= 0 \end{aligned}\right. \quad \text{for } i,j = 0,1. \qquad (3.7)$$

If the solution to the original *interpolation* problem is again denoted by U, we want to represent U as the sum of W and a yet to be determined function V:

$$U(x,y) = W(x,y) + V(x,y). \tag{3.8}$$

Now, since U is to satisfy the boundary conditions $L_i[U] = L_i[F]$ and $M_j[U] = M_j[F]$ $(i,j = 0,1)$, we have from (3.7) that:

$$L_i M_j[V] = M_j L_i[V] = 0 \qquad (i,j = 0,1), \tag{3.9}$$

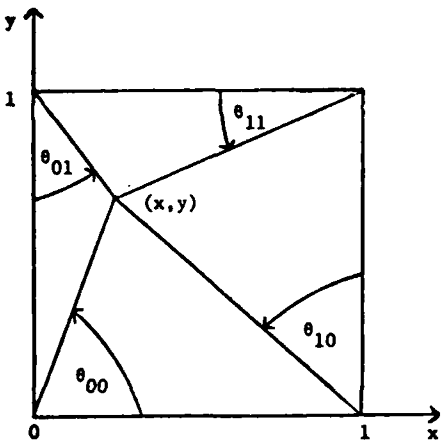
which is to say that the function V *satisfies consistent boundary conditions*, as defined in relation (2.2). Therefore, we can actually construct V using the techniques presented in the previous section. Referring back to (2.7), we have that:

$$\begin{aligned} V(x,y) &= (P_x \oplus P_y)[F - W] \\ &= P_x[F - W] + P_y[F - W], \end{aligned} \tag{3.10}$$

the last because of (3.7).

In summary, we first construct the function U_{ij} for each corner (i,j). Then, we compute the derived boundary conditions, $L_i[F - W]$ and $M_j[F - W]$, and use these in expression (3.10) for V. The function $U = W + V$ will then exactly satisfy the *original, inconsistent boundary conditions:* $L_i[U] = L_i[F]$ and $M_j[U] = M_j[F]$ for $i,j = 0,1$.

We shall now without derivation, display the functions U_{ij}. (For a more complete treatment, see [7].) For every point (x,y) in $S = [0,1] \times [0,1]$, define the angles θ_{ij} $(i,j = 0,1)$ indicated in the accompanying figure:



$$\begin{aligned} \theta_{00} &= \arctan(\tfrac{y}{x}) & \theta_{10} &= \arctan(\tfrac{1 - x}{y}) \\ \theta_{01} &= \arctan(\tfrac{x}{1 - y}) & \theta_{11} &= \arctan(\tfrac{1 - y}{1 - x}). \end{aligned} \tag{3.11}$$

We then define the functions U_{ij} as follows:

$$\begin{aligned} U_{00}(x,y) &= \phi_0(x)\psi_0(y)[T(\theta_{00})L_0M_0[F] \\ &\quad + (1-T(\theta_{00}))M_0L_0[F]] \\ U_{01}(x,y) &= \phi_0(x)\psi_1(y)[T(\theta_{01})M_1L_0[F] \\ &\quad + (1-T(\theta_{01}))L_0M_1[F]] \\ U_{10}(x,y) &= \phi_1(x)\psi_0(y)[T(\theta_{10})M_0L_1[F] \\ &\quad + (1-T(\theta_{10}))L_0M_1[F]] \\ U_{11}(x,y) &= \phi_1(x)\psi_1(y)[T(\theta_{11})L_1M_1[F] \\ &\quad + (1-T(\theta_{11}))M_1L_1[F]]. \end{aligned} \tag{3.12}$$

The functions $\phi_i(x)$ and $\psi_j(y)$ $(i,j = 0,1)$ are the same as in (2.6), and the $T(\theta_{ij})$ must satisfy the following conditions:

$$T(\theta_{ij}) = 1 \text{ at } \theta_{ij} = 0, \quad T(\theta_{ij}) = 0 \text{ at } \theta_{ij} = \frac{\pi}{2} \tag{3.13a}$$

$$\frac{\partial}{\partial\theta_{ij}} T(\theta_{ij}) = 0 \text{ at } \theta_{ij} = 0, \quad \frac{\partial}{\partial\theta_{ij}} T(\theta_{ij}) = 0 \text{ at } \theta_{ij} = \frac{\pi}{2} \tag{3.13b}$$

$$\frac{\partial^2}{\partial\theta_{ij}^2} T(\theta_{ij}) = 0 \text{ at } \theta_{ij} = \frac{\pi}{4}. \tag{3.14}$$

In the case of Dirichlet boundary conditions, only equations (3.13a) must hold, and they are quite simply satisfied by taking:

$$T(\theta_{ij}) = (1 - \frac{2\theta_{ij}}{\pi}) \qquad i,j = 0,1. \tag{3.15}$$

For the more general operators L_i and M_j, the cubic function

$$T(\theta_{ij}) = (\frac{2\theta_{ij}}{\pi} - 1)^2(\frac{4\theta_{ij}}{\pi} + 1) \qquad (i,j = 0,1) \tag{3.16}$$

satisfies (3.13a), (3.13b) and (3.14) as required.

Figure 5a shows a perspective view and 5b the contour plot of the interpolant to the mixed inconsistent boundary conditions:

$$\begin{aligned} L_0[F] &= F(0,y) &&= \cosh(\tfrac{\pi}{2}(1-y)) + 1 \\ L_1[F] &= F(1,y) + F_x(1,y) &&= \cosh(\tfrac{\pi}{2}(1-y))\sin(\tfrac{\pi}{2}\, y) \\ M_0[F] &= F(x,0) &&= .5x^2 + 1 \\ M_1[F] &= F_y(x,1) &&= 0 \end{aligned} \tag{3.17}$$

where:

$$\begin{aligned} \phi_0(x) &= 1 - .5x & \phi_1(x) &= .5x \\ \psi_0(y) &= 1 & \psi_1(y) &= y \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} L_0M_0[F] &= 1 & L_1M_0[F] &= 2.5 \\ M_0L_0[F] &= 3.509 & M_0L_1[F] &= 0 \\ L_0M_1[F] &= 0 & L_1M_1[F] &= 0 \\ M_1L_0[F] &= 0 & M_1L_1[F] &= 0. \end{aligned} \tag{3.19}$$

Figure 6a shows a perspective and 6b the contour plot of the interpolant to the mixed inconsistent boundary conditions:

$$\begin{aligned} L_0[F] &= F(0,y) = .25\sin(\pi(4y+.5)) + .7 \\ L_1[F] &= F(1,y) = 2y(1-y)\cos(\pi(2y-.25)) + .2 \end{aligned} \tag{3.20}$$

$M_0[F] = F_y(x,0) = 5(1-x)$

$M_1[F] = F_y(x,1) = 0$

where:

$$\phi_0(x) = 1 - x \qquad \phi_1(x) = x$$
$$\psi_0(y) = x - .5x^2 \qquad \psi_1(y) = .5x^2 \tag{3.21}$$

and

$$L_0M_0[F] = 5 \qquad L_1M_0[F] = 0$$
$$M_0L_0[F] = 0 \qquad M_0L_1[F] = 1.414$$
$$L_0M_1[F] = 0 \qquad L_1M_1[F] = 0 \tag{3.22}$$
$$M_1L_0[F] = 0 \qquad M_1L_1[F] = -1.414.$$

Note that although the boundary conditions are inconsistent at the three corners (0,0), (1,0) and (1,1), the function value of the interpolant is inconsistent only at the corner (1,1).

To illustrate the construction of $U(x,y)$ from $W(x,y)$ and $V(x,y)$ we will consider a very simple problem with Dirichlet boundary conditions and a discontinuity at (1,1):

$$L_0[F] = F(0,y) = y^2 \qquad L_1[F] = F(1,y) = 0$$
$$M_0[F] = F(x,0) = 0 \qquad M_1[F] = F(x,1) = 0. \tag{3.23}$$

Obviously, $L_iM_j[F] = M_jL_i[F]$ for $i = 0$ and $j = 0,1$. But, $L_0M_1[F] = 0$ and $M_1L_0[F] = 1$. The blending functions are:

$$\phi_0(x) = 1 - x \qquad \phi_1(x) = x$$
$$\psi_0(y) = 1 - y \qquad \psi_1(y) = y, \tag{3.24}$$

which yield

$$U(x,y) = (1-x)[y(1 - \frac{2\theta_{01}}{\pi}) - y + y^2]$$
$$= (1-x)y(y - \frac{2\theta_{01}}{\pi}) \tag{3.25}$$

where θ_{01} is defined in (3.11).

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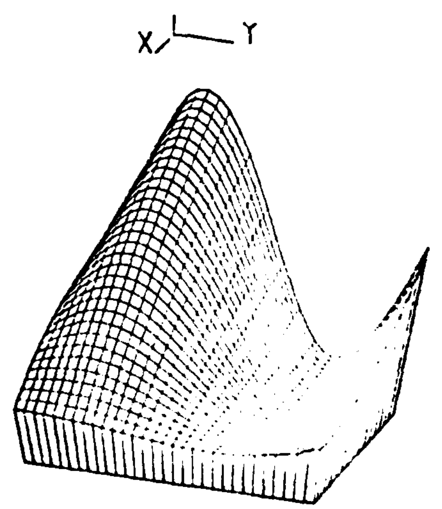

Figure 1a.

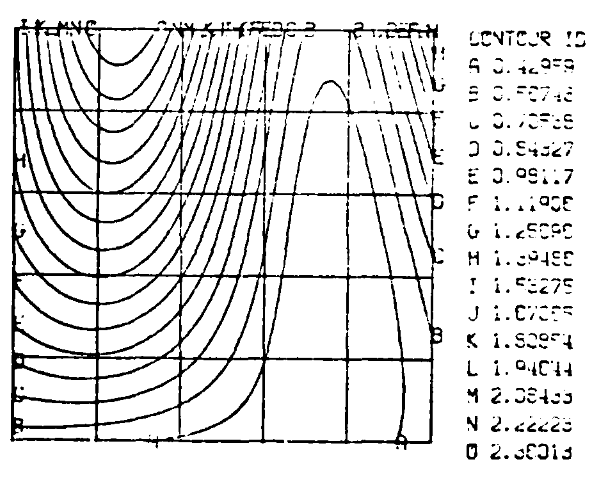

Figure 1b.

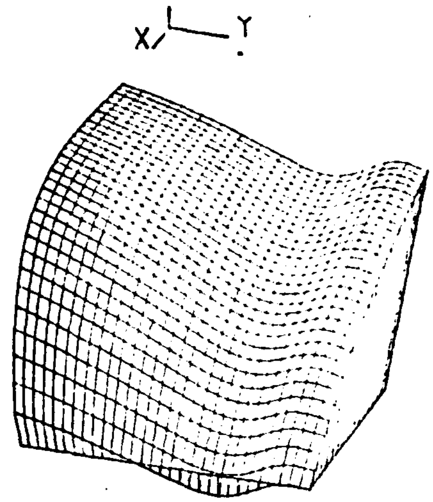

Figure 2a.

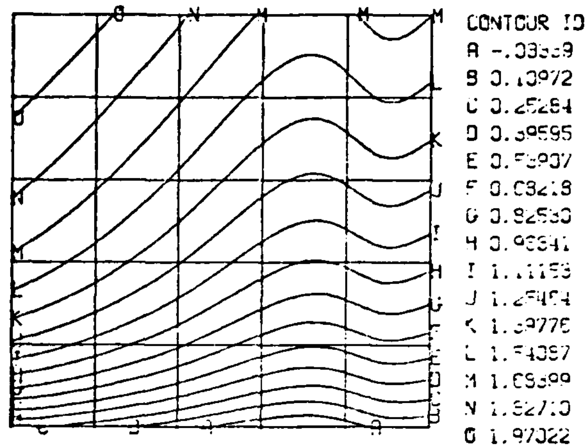

Figure 2b.

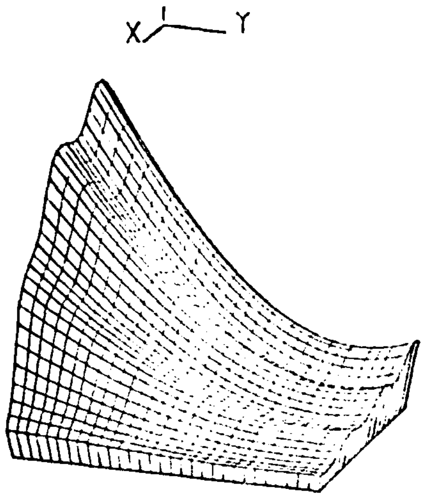

Figure 3a.

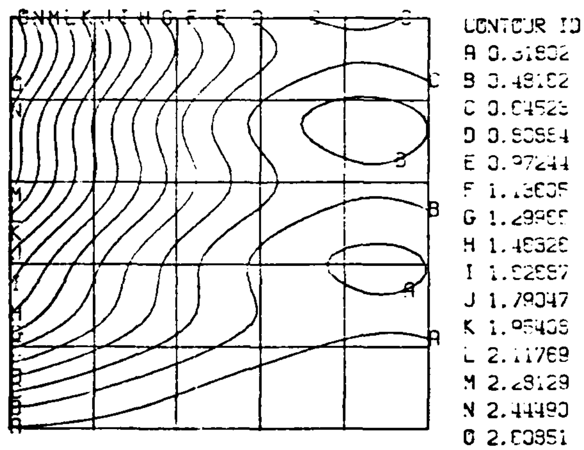

Figure 3b.

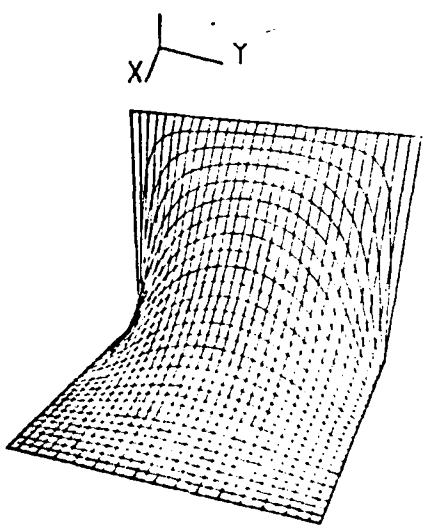

Figure 4a.

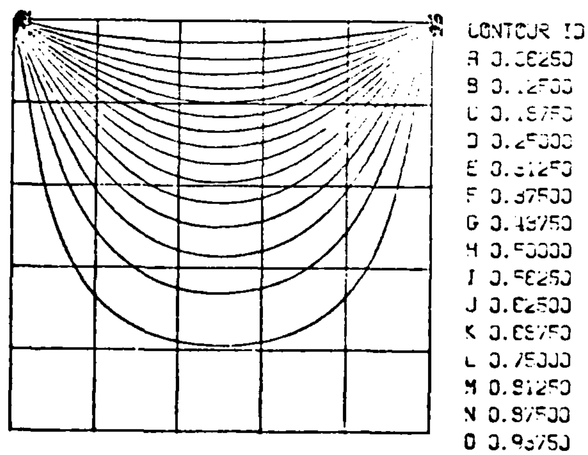

Figure 4b.

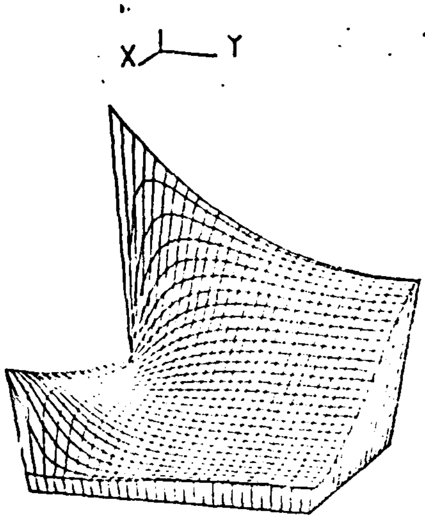

Figure 5a.

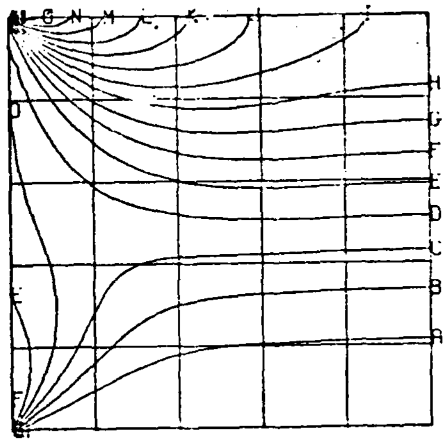

Figure 5b.

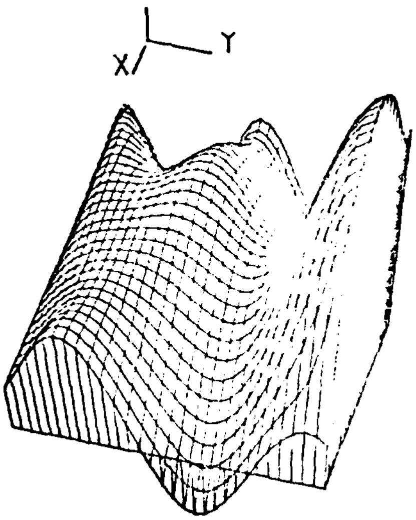

Figure 6a.

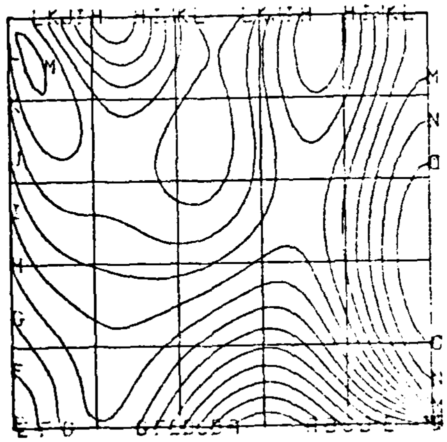

Figure 6b.

EXACT A PRIORI MATCHING OF MIXED BOUNDARY CONDITIONS
FOR SECOND ORDER ELLIPTIC PROBLEMS

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ERRATA

1. p. 2, equation (2.7), $(P_x + P_y)[F]$ should be $(P_x \oplus P_y)[F]$.

2. p. 3, equation (2.11), $\left[(y-1)\ \ln\sqrt{(x-1.2)^2 + .81} - \frac{1.8}{(x-1.2)^2 + .81}\right]$

should be $(y-1)\left[\ln\sqrt{(x-1.2)^2 + .81} - \frac{1.8}{(x-1.2)^2 + .81}\right]$

3. p. 5, equation (3.20), $L_1[F] = F(1,y) = 2y(1-y)\cos(\pi(2y-.25) + .2$

should be $L_1[F] = F(1,y) = 2y(1-y)\cos(\pi(2y-.25)) + .2$

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